**Solution Manual for**
**A First Course in Abstract Algebra, with Applications**
**Third Edition**
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# Exercises for Chapter 1

**1.1** True or false with reasons.

**(i)** There is a largest integer in every nonempty set of negative integers.

**Solution.** True. If $C$ is a nonempty set of negative integers, then

$$-C = \{-n : n \in C\}$$

is a nonempty set of positive integers. If $-a$ is the smallest element of $-C$, which exists by the Least Integer Axiom, then $-a \leq -c$ for all $c \in C$, so that $a \geq c$ for all $c \in C$.

**(ii)** There is a sequence of 13 consecutive natural numbers containing exactly 2 primes.

**Solution.** True. The integers 48 through 60 form such a sequence; only 53 and 59 are primes.

**(iii)** There are at least two primes in any sequence of 7 consecutive natural numbers.

**Solution.** False. The integers 48 through 54 are 7 consecutive natural numbers, and only 53 is prime.

**(iv)** Of all the sequences of consecutive natural numbers not containing 2 primes, there is a sequence of shortest length.

**Solution.** True. The set $C$ consisting of the lengths of such (finite) sequences is a nonempty subset of the natural numbers.

**(v)** 79 is a prime.

**Solution.** True. $\sqrt{79} < \sqrt{81} = 9$, and 79 is not divisible by 2, 3, 5, or 7.

**(vi)** There exists a sequence of statements $S(1), S(2), \ldots$ with $S(2n)$ true for all $n \geq 1$ and with $S(2n-1)$ false for every $n \geq 1$.

**Solution.** True. Define $S(2n-1)$ to be the statement $n \neq n$, and define $S(2n)$ to be the statement $n = n$.

**(vii)** For all $n \geq 0$, we have $n \leq F_n$, where $F_n$ is the $n$th Fibonacci number.

**Solution.** True. We have $0 = F_0$, $1 = F_1$, $1 = F_2$, and $2 = F_3$. Use the second form of induction with base steps $n = 2$ and $n = 3$ (verifying the inductive step will show why we choose these numbers). By the inductive hypothesis, $n - 2 \le F_{n-2}$ and $n - 1 \le F_{n-1}$. Hence, $2n - 3 \le F_n$. But $n \le 2n - 3$ for all $n \ge 3$, as desired.

**(viii)** If $m$ and $n$ are natural numbers, then $(mn)! = m!n!$.

**Solution.** False. If $m = 2 = n$, then $(mn)! = 24$ and $m!n! = 4$.

**1.2** **(i)** For any $n \ge 0$ and any $r \ne 1$, prove that

$$1 + r + r^2 + r^3 + \cdots + r^n = (1 - r^{n+1})/(1 - r).$$

**Solution.** We use induction on $n \ge 1$. When $n = 1$, both sides equal $1 + r$. For the inductive step, note that

$$\begin{aligned}[1 + r + r^2 + r^3 + \cdots + r^n] + r^{n+1} &= (1 - r^{n+1})/(1 - r) + r^{n+1} \\ &= \frac{1 - r^{n+1} + (1 - r)r^{n+1}}{1 - r} \\ &= \frac{1 - r^{n+2}}{1 - r}.\end{aligned}$$

**(ii)** Prove that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

**Solution.** This is the special case of the geometric series when $r = 2$; hence, the sum is $(1 - 2^{n+1})/(1 - 2) = 2^{n+1} - 1$. One can also prove this directly, by induction on $n \ge 0$.

**1.3** Show, for all $n \ge 1$, that $10^n$ leaves remainder 1 after dividing by 9.

**Solution.** This may be rephrased to say that there is an integer $q_n$ with $10^n = 9q_n + 1$. If we define $q_1 = 1$, then $10 = q_1 + 1$, and so the base step is true.

For the inductive step, there is an integer $q_n$ with

$$\begin{aligned}10^{n+1} = 10 \times 10^n &= 10(9q_n + 1) \\ &= 90q_n + 10 = 9(10q_n + 1) + 1.\end{aligned}$$

Define $q_{n+1} = 10q_n + 1$, which is an integer.

**1.4** Prove that if $0 \le a \le b$, then $a^n \le b^n$ for all $n \ge 0$.

**Solution.** *Base step.* $a^0 = 1 = b^0$, and so $a^0 \le b^0$.

*Inductive step.* The inductive hypothesis is

$$a^n \le b^n.$$

Since $a$ is positive, Theorem 1.4(i) gives $a^{n+1} = aa^n \leq ab^n$; since $b$ is positive, Theorem 1.4(i) now gives $ab^n \leq bb^n = b^{n+1}$.

**1.5** Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$.
**Solution.** The proof is by induction on $n \geq 1$. When $n = 1$, the left side is 1 and the right side is $\frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1$.

For the inductive step,

$$[1^2 + 2^2 + \cdots + n^2] + (n+1)^2 = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n + (n+1)^2$$
$$= \tfrac{1}{3}(n+1)^3 + \tfrac{1}{2}(n+1)^2 + \tfrac{1}{6}(n+1),$$

after some elementary algebraic manipulation.

**1.6** Prove that $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$.
**Solution.** *Base step*: When $n = 1$, both sides equal 1.

*Inductive step*:

$$[1^3 + 2^3 + \cdots + n^3] + (n+1)^3 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2 + (n+1)^3.$$

Expanding gives

$$\tfrac{1}{4}n^4 + \tfrac{3}{2}n^3 + \tfrac{13}{4}n^2 + 3n + 1,$$

which is

$$\tfrac{1}{4}(n+1)^4 + \tfrac{1}{2}(n+1)^3 + \tfrac{1}{4}(n+1)^2.$$

**1.7** Prove that $1^4 + 2^4 + \cdots + n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$.
**Solution.** The proof is by induction on $n \geq 1$. If $n - 1$, then the left side is 1, while the right side is $\frac{1}{5} + \frac{1}{2} + \frac{1}{3} - \frac{1}{30} = 1$ as well.

For the inductive step,

$$\left[1^4 + 2^4 + \cdots + n^4\right] + (n+1)^4 = \tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n + (n+1)^4.$$

It is now straightforward to check that this last expression is equal to

$$\tfrac{1}{5}(n+1)^5 + \tfrac{1}{2}(n+1)^4 + \tfrac{1}{3}(n+1)^3 - \tfrac{1}{30}(n+1).$$

**1.8** Find a formula for $1+3+5+\cdots+(2n-1)$, and use mathematical induction to prove that your formula is correct.
**Solution.** We prove by induction on $n \geq 1$ that the sum is $n^2$.

*Base Step.* When $n = 1$, we interpret the left side to mean 1. Of course, $1^2 = 1$, and so the base step is true.

*Inductive Step.*

$$1 + 3 + 5 + \cdots + (2n-1) + (2n+1)$$
$$= 1 + 3 + 5 + \cdots + (2n-1)] + (2n+1)$$
$$= n^2 + 2n + 1$$
$$= (n+1)^2.$$